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SOME REMARKS ON THE DARÓCZY EQUATION

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Abstract. The general solution of the functional equation

$$f(x) = f(x+1) + f(x(x+1)),$$

considered both on $(0, +\infty)$ and $\mathbb{R}$, are studied. Constructions of odd and even solutions are given.

In this paper we deal with the functional equation

$$f(x) = f(x+1) + f(x(x+1)) \tag{1}$$

and its real solution, generally defined on $(0, +\infty)$. Some problems concerning this equation was posed by Z.Daróczy during the XXIV ISFE in South Hadley [3]. The main problem was solved by M.Laczkovich and R.Redheffer [5]; see also [6], [1], [2], [4]. In part 1 we investigate the general solution $f : (0, +\infty) \to \mathbb{R}$ of (1) in the spirit of [6] by Z.Moszner. Next we give another construction of the general solution of the Daróczy equation which bases on an equivalence relation on $(0, +\infty)$. In part 3 we present constructions of real solutions of equation (1) defined on $\mathbb{R}$. In particular, we construct of all the odd and all the even solutions of (1). Finally, in part 4 we introduce another equation, a generalization of (1), and give some informations on its solutions under the assumption that there exists the limit $\lim_{x \to +\infty} xf(x)$, like it is in papers of K. Baron [1], [2] and W. Jarczyk [4].

1. Let us start with a simple remark: putting x instead of $x(x+1)$ in (1) we obtain

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REMARK 1. A function $f\colon (0,+\infty) \to \mathbb{R}$ is a solution of (1) if and only if

$$f(x) = f\left(\frac{\sqrt{1+4x}-1}{2}\right) - f\left(\frac{\sqrt{1+4x}+1}{2}\right) \tag{2}$$

for $x \in (0,+\infty)$.

The following theorem brings a description of the general solution of (1). In a special case ($a = 6$) it reduces to the result of Z.Moszner [6].

THEOREM 1. *If $a \in (2,6]$ then for every real function f_0 defined on $[\frac{\sqrt{1+4a}-1}{2}, a)$ there exists exactly one solution $f\colon (0,+\infty) \to \mathbb{R}$ of (1) which is an extension of f_0.*

PROOF. Define $\varphi : [0,+\infty) \to \mathbb{R}$ by

$$\varphi(x) := \frac{\sqrt{1+4x}-1}{2}, \tag{3}$$

observe that

$$\begin{aligned} &0 < \varphi(x) < x && \text{for} \quad x \in (0,+\infty), \quad \varphi(0) = 0,\\ &\varphi^{-1}(x-1) > x && \text{for} \quad x \in (2,+\infty) \end{aligned}$$

and let $(a_n : n \in \mathbb{Z})$, $(b_n : n \in \mathbb{N})$ be the sequences such that

$$\begin{aligned} a_0 =&\varphi(a) \quad \text{and} \quad \varphi(a_n) = a_{n-1} \quad \text{for} \quad n \in \mathbb{Z},\\ b_0 =&a \quad \text{and} \quad b_n = \varphi^{-1}(b_{n-1} - 1) \quad \text{for} \quad n \in \mathbb{N}. \end{aligned}$$

The sequence $(b_n : n \in \mathbb{N})$ is strictly increasing to infinity. Hence we can find the number $N \in \mathbb{N}$ such that

$$b_{N-1} < a + 1 \qquad \text{and} \quad b_N \geq a + 1.$$

Then

$$a_1 = a < b_N = \varphi^{-1}(b_{N-1} - 1) < \varphi^{-1}(a) = \varphi^{-1}(a_1) = a_2.$$

Define now functions $f_{1,1}, f_{1,2}, \dots, f_{1,N+1}$ in the following way:

$$\begin{aligned} &f_{1,1}(x) := f_0(\varphi(x)) - f_0(\varphi(x)+1), && x \in [a_1, b_1),\\ &f_{1,n}(x) := f_0(\varphi(x)) - f_{1,n-1}(\varphi(x)+1), && x \in [b_{n-1}, b_n), n = 2, \dots, N,\\ &f_{1,N+1}(x) := f_0(\varphi(x)) - f_{1,N}(\varphi(x)+1), && x \in [b_N, a_2), \end{aligned}$$

and put

$$f_1 := \bigcup_{j=1}^{N+1} f_{1,j}.$$

Also the sequence $(a_n : n \in \mathbb{Z})$ is strictly increasing and $\lim_{n\to-\infty} a_n = 0$, $\lim_{n\to+\infty} a_n = +\infty$. For every positive integer $n \geq 2$ define the function $f_n : [a_n, a_{n+1}) \to \mathbb{R}$ by putting

$$f_n(x) := \begin{cases} f_{n,1}(x), & x \in [a_n, \varphi^{-1}(a_n - 1)), \\ f_{n,2}(x), & x \in [\varphi^{-1}(a_n - 1), a_{n+1}), \end{cases}$$

where

$$f_{n,1}(x) := f_{n-1}(\varphi(x)) - f_{n-1}(\varphi(x)+1), \qquad x \in [a_n, \varphi^{-1}(a_n - 1)),$$

$$f_{n,2}(x) := f_{n-1}(\varphi(x)) - f_{1,n}(\varphi(x)+1), \qquad x \in [\varphi^{-1}(a_n - 1), a_{n+1}).$$

To define $f_n : [a_n, a_{n+1}) \to \mathbb{R}$ for negative integers we put

$$f_{-1}(x) := f_0(x+1) + f_0(x(x+1)) \qquad \text{for} \quad x \in [a_{-1}, a_0),$$

$$f_{n-1}(x) := \begin{cases} f_0(x+1) + f_n(x(x+1)), & x \in [a_{n-1}, a_n) \cap [a_0 - 1, a_1 - 1), \\ f_{-1}(x+1) + f_n(x(x+1)), & x \in [a_{n-1}, a_n) \cap [a_{-1} - 1, a_0 - 1), \end{cases}$$

for $n \leq -1$. Finally we define $f : (0, +\infty) \to \mathbb{R}$ by

$$f(x) := f_n(x) \qquad \text{for} \quad x \in [a_n, a_{n+1}), \quad n \in \mathbb{Z}.$$

It follows from the definition of f_n for $n \geq 1$ that (2) holds for $x \geq a_1$, whereas the definition of f_n for $n \leq -1$ gives (1) for positive $x < a_0$. Hence, since $x \leq a_1$ implies $\varphi(x) < a_0$, we have

$$f(\varphi(x)) = f(\varphi(x)+1) + f(x) \qquad \text{for} \quad x \in (a_0, a_1).$$

In other words, f is a solution of (2). According to Remark 1 it is also a solution of (1).

Finally, if $\tilde{f} : (0, +\infty) \to \mathbb{R}$ is a solution of (1) and an extension of f_0 then $f_n(x) = \tilde{f}(x)$ for $x \in [a_n, a_{n+1})$ and $n \in \mathbb{Z}$ whence $f = \tilde{f}$. □

COROLLARY 1. *If two solutions of* (1) *defined on* $(0, +\infty)$ *coincides on* $[\frac{\sqrt{1+4a}-1}{2}, a)$ *for some* $a \in (2, 6]$, *then they are identical.*

Later (in Remark 2 below) we shall show that the above theorem doesn't hold for $a = 2$. However, we have the following result.

THEOREM 2. *Let $f_1, f_2 : (0, +\infty) \to \mathbb{R}$ are solutions of* (1) *such that either*

(i) *there exist the limits*

$$\lim_{x \to 2+} f_1(x), \qquad \lim_{x \to 2+} f_2(x),$$

and at least one of them is finite;
or

(ii) *there exists an $\varepsilon > 0$ such that*

$$f_1(x) \geq f_2(x) \qquad \text{for} \quad x \in (2, 2 + \varepsilon).$$

If

$$f_1 \,|_{[1,2)} = f_2 \,|_{[1,2)}$$

then

$$f_1 = f_2.$$

PROOF. Defining

$$f := f_1 - f_2$$

we observe that f is a solution of (1) vanishing on $[1, 2)$. We shall show that it vanishes on $[1, 6)$. Putting $x = 1$ in (1) we obtain $f(2) = 0$. Fix $x_0 \in (2, 6)$, define φ: $(0, +\infty) \to \mathbb{R}$ by (3) and the sequence $(x_n : n \in \mathrm{N})$ putting

$$x_n := \varphi(x_{n-1}) + 1.$$

We can easy show that this sequence is strictly decreasing to 2. In particular,

$$\varphi(x_n) \in \varphi((2, 6)) = (1, 2).$$

Hence

$$0 = f(\varphi(x_n)) = f(\varphi(x_n) + 1) + f(\varphi(x_n)(\varphi(x_n) + 1)) = f(x_{n+1}) + f(x_n)$$

i.e.

$$f(x_{n+1}) = -f(x_n) \qquad \text{for} \quad n \in \mathrm{N}_0.$$

This gives

$$f(x_n) = (-1)^n f(x_0) \qquad \text{for} \quad n \in \mathrm{N}.$$

In case (i) the sequence $(f(x_n) : n \in \mathbb{N})$ has a limit whence $f(x_0) = 0$. In case (ii) we have $f(x_n) \geq 0$ for n large enough and so $f(x_0) = 0$ as well. Thus we have proved that f vanishes on $(1,6)$ and it follows from Corollary 1 that f vanishes everywhere. It means that $f_1 = f_2$. □

Now we shall explain more precisely non-uniqueness in extending functions from $[1,2)$ to solutions of Daróczy equation on $(0,+\infty)$.

REMARK 2. For any solution $f_1 : (0,+\infty) \to \mathbb{R}$ of (1), for any $a \in (2,6]$ and for any function $u : [\frac{\sqrt{1+4a}+1}{2}, a) \to \mathbb{R}$ there exists a solution $f_2 : (0,+\infty) \to \mathbb{R}$ of (1) such that

$$f_1(x) = f_2(x) \qquad \text{for} \quad x \in (0,2]$$

and

$$f_1(x) - f_2(x) = u(x) \qquad \text{for} \quad x \in \left[\frac{\sqrt{1+4a}+1}{2}, a\right).$$

We precede our proof of this remark by the following lemma.

LEMMA 1. *If a solution of* (1) *on* $(0,+\infty)$ *vanishes on* $(1,2]$ *then it vanishes on* $(0,2]$.

PROOF. Let $f : (0,+\infty) \to \mathbb{R}$ be a solution of (1) vanishing on $(1,2]$. Define $\varphi: (0,+\infty) \to \mathbb{R}$ by (3) and the sequence $(x_n : n \in \mathbb{N})$ putting

$$x_0 := 2 \quad \text{and} \quad x_n := \varphi(x_{n-1}) \quad \text{for} \quad n \in \mathbb{N}.$$

This sequence is strictly decreasing to zero and $x_1 = 1$. Moreover, if $n \in \mathbb{N}$ and $x \in (x_{n+1}, x_n]$ then $x+1 \in (x_1, x_0]$ and $x(x+1) \in (x_n, x_{n-1}]$. Hence f vanishes on $(x_1, x_0]$ and if f vanishes on $(x_n, x_{n-1}]$ then,as a solution of (1), it vanishes also on $(x_{n+1}, x_n]$. □

PROOF OF REMARK 2. We have to define a solution $f: (0,+\infty) \to \mathbb{R}$ of (1) which vanishes on $(0,2]$ and coincides with u on $[\frac{\sqrt{1+4a}+1}{2}, a)$. Define $\psi: (2,+\infty) \to \mathbb{R}$ by

$$\psi(x) := \frac{\sqrt{1+4x}+1}{2}$$

and the sequence $(c_n : n \in \mathbb{N})$ putting

$$c_1 := a \qquad \text{and} \quad c_{n+1} := \psi(c_n) \quad \text{for} \quad n \in \mathbb{N}.$$

This sequence is strictly decreasing to 2. Hence for every $n \in \mathbf{N}$ we can define the function $f_n : [c_{n+1}, c_n) \to \mathbf{R}$ by

$$f_n(x) := (-1)^{n-1} u(\psi^{-(n-1)}(x)) \qquad \text{for} \quad x \in [c_{n+1}, c_n).$$

Putting

$$f_0(x) := \begin{cases} f_n(x), & x \in [c_{n+1}, c_n), : : n \in \mathbf{N}, \\ 0, & x \in [\dfrac{\sqrt{1+4a}-1}{2}, 2], \end{cases}$$

and using Theorem 1 we obtain a solution $f \colon (0, +\infty) \to \mathbf{R}$ of (1) which is an extension of f_0; in particular f coincides with u on $[\frac{\sqrt{1+4a}+1}{2}, a)$. Now we show that f vanishes on $(0, 2]$. On virtue of Lemma 1 and the definition of f_0 it is enough to check that f vanishes on $(1, \frac{\sqrt{1+4a}-1}{2})$. Let $x \in (1, \frac{\sqrt{1+4a}-1}{2})$. Then $x + 1 \in (2, c_2)$ and there exists an $n \geq 2$ such that $x + 1 \in [c_{n+1}, c_n)$. Hence

$$x(x+1) = \psi^{-1}(x+1) \in [\psi^{-1}(c_{n+1}), \psi^{-1}(c_n)) = [c_n, c_{n-1})$$

and

$$\begin{aligned} f(x) =& f(x+1) + f(x(x+1)) \\ =& (-1)^{n-1} u(\psi^{-(n-1)}(x+1)) + (-1)^{n-2} u(\psi^{-(n-2)}(x(x+1))) \\ =& (-1)^{n-1} u(\psi^{-(n-1)}(x+1)) + (-1)^{n} u(\psi^{-(n-2)}(\psi^{-1}(x+1))) \\ =& (-1)^{n-1} u(\psi^{-(n-1)}(x+1)) + (-1)^{n} u(\psi^{-(n-1)}(x+1)) = 0. \end{aligned}$$

□

2. In this section we present another construction of solutions of Daróczy equation and we give two examples of discontinuous at each point solutions: such that there exists the limit at infinity and such that this limit does not exist.

THEOREM 3. *There exists a partition $\mathcal{X}$ of $(0, +\infty)$ consisting of countable and dense subsets of $(0, +\infty)$ such that*

(4) $\quad$ *if*: $X \in \mathcal{X}$ *and* $x \in X$ *then* $x + 1,\ x(x+1) \in X$;

in particular, a function $f : (0, +\infty) \to \mathbf{R}$ is a solution of (1) iff for every $X \in \mathcal{X}$ the function $f|_X$ does.

PROOF. Define $\varphi \colon (0, +\infty) \to \mathbf{R}$ by (3) and $\tau \colon (0, +\infty) \to \mathbf{R}$ by

$$\tau(x) = x + 1,$$

put

$$\Phi = \{\varphi,\ \varphi^{-1},\ \tau,\ \tau^{-1}\}$$

and define the relation $\sim$ on $(0, +\infty)$ by

$$x \sim y \iff y = \varphi_1(\ldots(\varphi_n(x)\ldots) \text{ for some } \varphi_1, \ldots, \varphi_n \in \Phi.$$

One can easily check that it is an equivalence relation and thus defines a partition $\mathcal{X}$ of $(0, +\infty)$ consisting of its equivalence classes. It is clear that if $X \in \mathcal{X}$ then X is countable and (4) holds. We shall show that X is also dense in $(0, +\infty)$. Suppose for the contrary that there exist $a, b \in (0, +\infty)$ such that $a < b$ and $(a, b) \cap X = \emptyset$. Then

$$\emptyset = \varphi^{-1}((a, b)) \cap \varphi^{-1}(X) = (\varphi^{-1}(a), \varphi^{-1}(b)) \cap X$$

and so (by induction)

$$(\varphi^{-n}(a), \varphi^{-n}(b)) \cap X = \emptyset \qquad \text{for every} \quad n \in \dot{\mathbf{N}}.$$

Since $\varphi^{-1}(x) > x$ for $x \in (0, +\infty)$ and $(\varphi^{-1})'(x) \geq 2a + 1$ for $x \geq a$ we have

$$\varphi^{-(n+1)}(b) - \varphi^{-(n+1)}(a) \geq (2a + 1)(\varphi^{-n}(b) - \varphi^{-n}(a))$$

for every $n \in \mathbf{N}$, whence

$$\lim_{n \to +\infty} (\varphi^{-n}(b) - \varphi^{-n}(a)) = +\infty.$$

Consequently there exists an $n \in \mathbf{N}$ such that

$$\varphi^{-n}(b) - \varphi^{-n}(a) > 1.$$

Let $x \in X$ and fix an integer k such that

$$x + k \in (\varphi^{-n}(a), \varphi^{-n}(b)).$$

Then

$$\tau^k(x) = x + k \in (\varphi^{-n}(a), \varphi^{-n}(b)) \cap X,$$

a contradiction. □

Theorem 3 allows us to give some interesting examples.

REMARK 3. **(i)** There exists a solution $f : (0, +\infty) \to (0, +\infty)$ of (1) which is discontinuous at each point and such that the limit

(5) $$\lim_{x \to +\infty} f(x).$$

does not exist.

(ii) There exist a solution $f\colon (0,+\infty) \to (0,+\infty)$ of (1) which is discontinuous at each point and such that

$$\lim_{x\to+\infty} f(x) = 0. \tag{6}$$

PROOF. Let $\mathcal{X}$ be a partition of $(0,+\infty)$ with the properites mentioned in Theorem 3, fix a non-constant function $c\colon \mathcal{X} \to (0,+\infty)$ and define a solution $f\colon (0,+\infty) \to (0,+\infty)$ of (1) by

$$f(x) := \frac{c(X)}{x} \qquad \text{for} \quad x \in X, \quad X \in \mathcal{X}.$$

It is clear that f is discontinuous at each point. If c is bounded then (6) holds and we have (ii). Assume c is unbounded. We shall prove that limit (5) does not exists. For, let $(X_n : n \in \mathbf{N})$ be a sequence of elements of $\mathcal{X}$ with $\lim_{n\to+\infty} c(X_n) = +\infty$ and for every $n \in \mathbf{N}$ choose an $x_n \in (c(X_n), 2c(X_n)) \cap X_n$. Then $\lim_{n\to+\infty} x_n = +\infty$ and

$$f(x_n) > \frac{1}{2} \qquad \text{for} \quad n \in \mathbf{N}. \tag{7}$$

If the limit (5) existed we would have

$$\lim_{n\to+\infty} f(x_n) = \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} f\,|_X\,(x) = 0$$

for every $X \in \mathcal{X}$, a contradiction with (7). □

3. In this part of the paper we shall show a construction of all the solutions of (1) defined on $\mathbb{R}$. Let us start with two simple lemmas.

LEMMA 2. *If $g\colon \mathbb{R} \to \mathbb{R}$ is a solution of (1) then the function $G\colon\mathbb{R} \to \mathbb{R}$ defined by*

$$G(x) := g(x) + g(-x) \tag{8}$$

is periodic with period 1.

PROOF. Fix $x \in \mathbb{R}$. Then, according to (1),

$$g(-x-1) = g(-x) + g(x(x+1)) = g(-x) + [g(x) - g(x+1)]$$

i.e. $G(x+1) = G(x)$. $\square$

LEMMA 3. *Every solution* $g:(-1,+\infty) \to \mathbb{R}$ *of* (1) *has a unique extension to a solution* $f:\mathbb{R} \to \mathbb{R}$ *of* (1).

PROOF. Define $G:[0,1) \to \mathbb{R}$ by (8) and $f:\mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} g(x), & x \in (-1,+\infty), \\ G(\{x\}) - g(-x), & x \in (-\infty,-1], \end{cases}$$

where $\{x\}$ denotes the fractal part of x. Observe that for every $x \in (0,1)$ we have

$$\begin{aligned} G(\{-x\}) =& G(1-x) = g(1-x) + g(x-1) \\ =& [g(-x) - g(-x(-x+1))] + g(x-1) \\ =& g(-x) - g(x(x-1)) + g(x-1) \\ =& g(-x) - [g(x-1) - g(x)] + g(x-1) \\ =& g(x) + g(-x) = G(\{x\}) \end{aligned}$$

whence

$$G(\{-x\}) = G(\{x\}) \qquad \text{for} \quad x \in \mathbb{R}.$$

Now we shall show that f is a solution of (1). Of course (1) holds for $x \in (-1,+\infty)$. Assume now that $n \in \mathbb{N}$ and (1) holds for every $x \in (-n,+\infty)$. Then for $x \in (-n-1,-n]$ we have

$$\begin{aligned} f(x) =& G(\{-x\}) - f(-x) = G(\{-x-1\}) - f(-x) \\ =& f(x+1) + g(-x-1) - f(-x) \\ =& f(x+1) + g(-x) + g(-x(-x-1)) - f(-x) \\ =& f(x+1) + f(x(x+1)) \end{aligned}$$

and so f is a solution of (1). Finally, if $\tilde{f}$ is an extension g to a solution of (1) then applying Lemma 2 we see that

$$\tilde{f}(x) + \tilde{f}(-x) = \tilde{f}(\{x\}) + \tilde{f}(-\{x\}) = g(\{x\}) + g(-\{x\}) = G(\{x\})$$

for $x \in \mathbb{R}$, whence for $x \in (-\infty,-1]$ we obtain

$$f(x) = G(\{x\}) - g(-x) = \tilde{f}(x) + \tilde{f}(-x) - \tilde{f}(-x) = \tilde{f}(x)$$

which ends the proof. $\square$

THEOREM 4. *If $a \in (2,6]$ then for every real function f_0 defined on the set*

$$\left[-\frac{1}{2},-\frac{1}{4}\right) \cup \{0\} \cup \left[\frac{\sqrt{1+4a}-1}{2},2\right) \cup (2,a)$$

there exists exactly one solution $f: \mathbb{R} \to \mathbb{R}$ of (1) which is an extension of f_0.

PROOF. First of all let us observe that any solution of (1) defined on $[0,+\infty)$ vanishes at 1 and 2. Hence, extending f_0 onto $[\frac{\sqrt{1+4a}-1}{2},a)$ by putting $f_0(2) = 0$ and applying Theorem 1 we see that f_0 has a unique extension to a solution $\tilde{f}_0:(0,+\infty) \to \mathbb{R}$ of (1). Extend now $\tilde{f}_0$ onto $[0,+\infty)$ by putting $\tilde{f}_0(0) = f_0(0)$. Then $\tilde{f}_0$ is the unique extension of f_0 to a solution of (1) defined on $[0,+\infty)$. Define $\varphi : [-\frac{1}{4},0) \to [-\frac{1}{2},0)$ by (3) and the sequence $(x_n : n \in \mathbb{N}_0)$ putting

$$x_0 := -\frac{1}{2} \qquad \text{and} \quad x_n := \varphi^{-1}(x_{n-1}) \quad \text{for} \quad n \in \mathbb{N}.$$

This sequence strictly increases to zero. For every positive integer n define a function $f_n:[x_n,x_{n+1}) \to \mathbb{R}$ by

$$f_n(x) := f_{n-1}(\varphi(x)) - \tilde{f}_0(\varphi(x)+1), \qquad x \in [x_n,x_{n+1}).$$

The formula

$$\tilde{f}_1 := f_n(x) \qquad \text{for} \quad x \in [x_n,x_{n+1}) \quad \text{and} \quad n \in \mathbb{N}_0$$

defines a function $\tilde{f}_1 : [-\frac{1}{2},0) \to \mathbb{R}$. With the aid of $\tilde{f}_0$ and $\tilde{f}_1$ define $\tilde{f}_2 : (-1,-\frac{1}{2}) \to \mathbb{R}$ putting

$$\tilde{f}_2(x) := \tilde{f}_0(x+1) + \tilde{f}_1(x(x+1)).$$

Finally we define $\tilde{f}:(-1,+\infty) \to \mathbb{R}$ by

$$\tilde{f} := \tilde{f}_0 \cup \tilde{f}_1 \cup \tilde{f}_2.$$

It is easy to see that $\tilde{f}$ is the unique extension of f_0 to a solution of (1) defined on $(-1,+\infty)$. An application of Lemma 3 ends the proof. □

The following simple theorem describes even solution of (1).

THEOREM 5. *The only even solution of (1) on $\mathbb{R}$ is the zero function.*

PROOF. If $f : \mathbb{R} \to \mathbb{R}$ is an even solution of (1) then an application of Lemma 2 shows that f is periodic with period 1 and (1) gives

$$f(x(x+1)) = 0 \qquad \text{for} \quad x \in \mathbb{R}.$$

In particular, $f(x) = 0$ for $x \in [0, +\infty)$ and, as f is even, $f = 0$. □

All the odd solutions of equation (1) defined on $\mathbb{R}$ describes the following theorem.

THEOREM 6. *If $a \in (2, 6]$ then for every real function f_0 defined on the set*

$$\left(0, \frac{1}{2}\right) \cup \left[\frac{\sqrt{1+4a}+1}{2}, a\right)$$

there exists exactly one odd solution $f : \mathbb{R} \to \mathbb{R}$ of (1) which is an extension of f_0.

PROOF. It is easy to observe that the function $\tilde{f}_0 : (0, 1) \to \mathbb{R}$ given by

$$\tilde{f}_0(x) := \begin{cases} f_0(x), & x \in \left(0, \dfrac{1}{2}\right), \\ \dfrac{1}{2} f_0(\frac{1}{4}), & x = \dfrac{1}{2}, \\ f_0(x(1-x)) - f_0(1-x), & x \in \left(\dfrac{1}{2}, 1\right), \end{cases} \tag{9}$$

satisfies

$$\tilde{f}_0(x) + \tilde{f}_0(1-x) = \tilde{f}_0(x(1-x)) \qquad \text{for} \quad x \in (0, 1). \tag{10}$$

Define $\psi : (1, +\infty) \to \mathbb{R}$ by $\psi(x) = (x-1)x$ and $(x_n : n \in \mathbb{N}_0)$ by

$$x_0 := 1 \qquad \text{and} \quad x_{n+1} := \psi^{-1}(x_n) \quad \text{for} \quad n \in \mathbb{N}.$$

This is a strictly increasing sequence with the limit equal to 2. For every non-negative integer n define a function $g_n : [x_n, x_{n+1}) \to \mathbb{R}$ putting

$$g_0(x_0) := 0 \qquad \text{and} \quad g_0(x) := \tilde{f}_0(x-1) - \tilde{f}_0(\psi(x)), \quad x \in (x_0, x_1), \tag{11}$$

$$g_n(x) := \tilde{f}_0(x-1) - g_{n-1}(\psi(x)), \quad x \in [x_n, x_{n+1}), \quad n \in \mathbb{N}, \tag{12}$$

and a function $\tilde{f}_1 : [1,2) \to \mathbb{R}$ as

$$\tilde{f}_1 := g_0 \cup g_1 \cup g_2 \cup \dots .$$

Consider also a sequence $(a_n : n \in \mathbb{N}_0)$ such that

$$a_0 := a \quad \text{and} \quad a_{n+1} := \psi^{-1}(a_n)) \quad \text{for} \quad n \in \mathbb{N}.$$

This sequence strictly decreases to 2. For every positive integer n define a function $h_n : [a_n, a_{n-1}) \to \mathbb{R}$ putting

$$\begin{aligned} h_1(x) &:= f_0(x), \quad x \in [a_1, a_0), \\ h_n(x) &:= \tilde{f}_1(x-1) - h_{n-1}(\psi(x)), \quad x \in [a_n, a_{n-1}), \quad n \geq 2, \end{aligned} \tag{13}$$

and a function $\tilde{f}_2 : (2, a) \to \mathbb{R}$ as

$$\tilde{f}_2 := h_1 \cup h_2 \cup \dots .$$

Furthemore, let

$$f_1 := \tilde{f}_0 \cup \tilde{f}_1 \cup \tilde{f}_2$$

and extend f_1 onto $[0, a)$ assuming additionally

$$f_1(0) := 0, \quad f_1(2) := 0. \tag{14}$$

It follows from (11)–(14) that

$$f_1(x) = f_1(x-1) - f_1(\psi(x)) \qquad \text{for} \quad x \in (1, a),$$

i.e. f_1 satisfy (1) for $x \in (0, a-1)$. Applying Theorem 1 to the function f_1 restricted to $[\frac{\sqrt{1+4a}-1}{2}, a)$ we obtain exactly one solution $f_2 : (0, +\infty) \to \mathbb{R}$ of (1) which coincides with f_1 on $[\frac{\sqrt{1+4a}-1}{2}, a)$. As the function

$$f_1 |_{(0,a)} \cup f_2 |_{[a,+\infty)} \tag{15}$$

coincides with f_1 on $[\frac{\sqrt{1+4a}-1}{2}, a)$ and is a solution of (1) it follows that (from Corollary 1) that the function (15) equals f_2. In particular f_2 is an extension of f_1. Consequently f_2 is an extension of f_0, $f_2(1) = 0$ and (cf. (10))

$$f_2(x) + f_2(1-x) = f_2(x(1-x)) \qquad \text{for} \quad x \in (0, 1). \tag{16}$$

Let $f:\mathbf{R}\to\mathbf{R}$ be the odd extension of f_2. We shall check that f is a solution of (1). Of course (1) holds for $x\in[0,+\infty)$. If $x\in(-\infty,-1)$, then

$$\begin{aligned}f(x+1)+f(x(x+1)) &= -f_2(-x-1)+f_2(x(x+1))\\ &= -f_2(-x-1)+f_2((-x-1)((-x-1)+1))\\ &= -f_2(-x-1)+f_2(-x-1)-f_2(-x)=f(x).\end{aligned}$$

Next, if $x\in(-1,0)$ then using (16) we have

$$f(x+1)+f(x(x+1))=f_2(x+1)-f_2(-x(x+1))=-f_2(-x)=f(x).$$

Finally, since $f(-1)=-f(1)=0$ we see that (1) holds for $x=-1$ as well.

To end the proof, assume that $\tilde f:\mathbf{R}\to\mathbf{R}$ is an odd solution of (1) and an extension of f_0. It follows from (9) that $\tilde f\mid_{(0,1)}=\tilde f_0$ whereas (1) gives $\tilde f(1)=0$. Hence and from (11) and (12) it follows that $\tilde f\mid_{[1,2)}=\tilde f_1$. This jointly with (13) shows that $\tilde f\mid_{(2,a)}=\tilde f_2$. Since (1) gives $\tilde f(2)=0$ we have $\tilde f\mid_{[0,a)}=f_1$. Applying now Theorem 1 we obtain $\tilde f\mid_{(0,+\infty)}=f_2$ and $\tilde f=f$. □

4. Fix a positive real number a. Of course,

$$\frac{1}{x}=\frac{a}{x(x+a)}+\frac{1}{x+a}\qquad\text{for}\quad x\in(0,+\infty).$$

In the other words, the function $f\colon(0,+\infty)\to\mathbf{R}$ defined by $f(x):=1/x$ is a solution of

$$f(x)=f(x+a)+f\left(\frac{x(x+a)}{a}\right)\tag{17}$$

as well of

$$f(x)=f(x+a)+af(x(x+a)).\tag{18}$$

In the case where $a=1$ each of these two equations reduce to (1). In fact (17) is equivalent to (1) for every $a>0$. For, if $f\colon(0,+\infty)\to\mathbf{R}$ is a solution of (1) then

$$f\left(\frac{x}{a}\right)=f\left(\frac{x}{a}+1\right)+f\left(\frac{x}{a}\left(\frac{x}{a}+1\right)\right)\qquad\text{for}\quad x\in(0,+\infty)$$

and putting $\tilde f(x):=f(x/a)$ we obtain

$$\tilde f(x)=\tilde f(x+a)+\tilde f\left(\frac{x(x+a)}{a}\right)\qquad\text{for}\quad x\in(0,+\infty),$$

i.e. $\tilde{f}$ is a solution of (17). However, as it follows from Theorem 8 below, in general equations (18) and (1) are not equivalent.

In this part of the paper we shall examine solutions of (18) under the assumption that there exists the limit $\lim_{x\to+\infty} xf(x)$ (see Baron [1], [2]) and we obtain solutions of (18) which are not of the form $\frac{c}{x}$ on the whole interval $(0,+\infty)$.

THEOREM 7. *Let $a \in (0,+\infty)$. If $f\colon (0,+\infty) \to \mathbb{R}$ is a solution of* (18) *such that there exists the limit*

$$\lim_{x\to+\infty} xf(x), \tag{19}$$

then this limit is finite and

$$f(x) = \frac{c}{x} \qquad \textit{for} \quad x \in (0,+\infty) \cap [1-a,+\infty)$$

with c being the limit (19).

Similarly as K. Baron did in [2], let us start with the following lemma.

LEMMA 4. *Let $a \in (0,+\infty)$ and $f\colon (0,+\infty) \to \mathbb{R}$ be a solution of* (18). *If there exists an $M > 0$ such that for some $c \in \mathbb{R}$ we have*

$$f(x) \le \frac{c}{x} \qquad \text{for} \quad x > M$$

then

$$f(x) \le \frac{c}{x} \qquad \text{for} \quad x \in (0,+\infty) \cap [1-a,+\infty).$$

PROOF. Replacing f by $\tilde{f}(x) = f(x) - c/x$, $x > 0$, we may assume that $c = 0$. Fix arbitrarily $x_0 \in (0,+\infty) \cap (1-a,+\infty)$ and define the sequence $(x_n : n \in \mathrm{N})$ by

$$x_{n+1} := \min\{x_n + a, x_n(x_n + a)\} \qquad \text{for} \quad n \in \mathrm{N}$$

It is easy to see that the sequence $(x_n : n \in \mathrm{N})$ increases to infinity. Using induction and (18) one can see that for every positive integer n there exists a sequence

$$(l_1, \dots, l_{2^n})$$

of non-negative integers and a sequence

$$(\alpha_1, \dots, \alpha_{2^n})$$

of numbers not smaller than x_n such that

$$(20) \qquad f(x_0) = \sum_{i=1}^{2^n} a^{l_i} f(\alpha_i).$$

Now, if n is a positive integer such that $x_n > M$ then (20) gives $f(x_0) \le 0$. This proves that f is nonpositive on $(0, +\infty) \cap (1 - a, +\infty)$. If $1 - a > 0$ then applying (18) we obtain that also $f(1 - a) \le 0$. □

PROOF OF THEOREM 7. When having Lemma 4, our Theorem 7 may be proved as the main result of [2]. For the sake of completeness we repeat this proof here.

Assume the limit (19) equals $-\infty$ and fix arbitrarily a real number c. Then there exists an $M > 0$ such that

$$xf(x) \le c \qquad \text{for} \quad x > M.$$

Hence and from the lemma we obtain

$$xf(x) \le c \qquad \text{for} \quad x \in (0, +\infty) \cap (1 - a, +\infty),$$

which leads to a contradiction as c was fixed arbitriarily. The case when the limit (19) equals $+\infty$ reduces to the previous one by considering the function $-f$. Up to now we have proved that the limit (19) is finite. Denote it by c and fix arbitriarily an $\varepsilon > 0$. Then there exists an $M > 0$ such that

$$xf(x) \le c + \varepsilon \qquad \text{for} \quad x > M.$$

Hence and from the lemma we obtain

$$xf(x) \le c + \varepsilon \qquad \text{for} \quad x \in (0, +\infty) \cap (1 - a, +\infty).$$

Consequently, as the positive number ε has been fixed arbitriarily we have

$$xf(x) \le c \qquad \text{for} \quad x \in (0, +\infty) \cap (1 - a, +\infty).$$

Applying it to the function $-f$ we shall obtain the reverse inequality which ends the proof. □

THEOREM 8. *If $a \in (0,1)$, $x_0 \in [1-2a, 1-a) \cap (0,1)$ and $x_1 := \frac{\sqrt{a^2+4x_0}-a}{2}$ then for every $c \in \mathbb{R}$ and for every $u : [x_0, x_1) \to \mathbb{R}$ there exists exactly one solution $f : (0, +\infty) \to \mathbb{R}$ of (18) which is an extension of u and*

$$(21) \qquad \lim_{x \to +\infty} xf(x) = c;$$

moreover, f is continuous iff u is continuous and

$$\lim_{x\to x_1} u(x) = au(x_0) + \frac{c}{x_1+a}. \tag{22}$$

PROOF. As in the proof of Lemma 4 we may assume that $c=0$. Define $\varphi\colon(0,+\infty)\to\mathbf{R}$ by

$$\varphi(x) := \frac{\sqrt{a^2+4x}-a}{2}.$$

Putting $\varphi(x)$ instead of x in (18) we obtain that $f\colon(0,+\infty)\to\mathbf{R}$ is a solution of (18) if and only if it is a solution of

$$f(x) = a^{-1}f(\varphi(x)) - a^{-1}f(\varphi(x)+a). \tag{23}$$

Let $(x_n : n\in\mathbf{Z})$ be the sequence such that

$$x_{n+1} = \varphi(x_n) \qquad \text{for} \quad n\in\mathbf{Z}.$$

Of course it is strictly increasing and $\lim_{n\to-\infty} x_n = 0$, $\lim_{n\to+\infty} x_n = 1-a$. Given $u\colon[x_0,x_1)\to\mathbf{R}$ define a function $f_0\colon[x_0,+\infty)\to\mathbf{R}$ by

$$f_0(x) := \begin{cases} a^n u(\varphi^{-n}(x)), & x\in[x_n,x_{n+1}), n\in\mathrm{N}_0,\\ 0, & x\in[1-a,+\infty).\end{cases}$$

Clearly, f_0 is an extension of u. We shall proof that f_0 is a solution of (23). It is obvious that (23) holds for $x\in[1-a,+\infty)$. Let $x\in[x_0,1-a)$. Then there exists an $n\in\mathrm{N}_0$ such that $x\in[x_n,x_{n+1})$ and

$$\varphi(x)\in\varphi([x_n,x_{n+1})) = [x_{n+1},x_{n+2}).$$

Since $x\geq x_0\geq 1-2a$, we have $\varphi(x)+a\geq 1-a$ and $f_0(\varphi(x)+a)=0$. Consequently,

$$\begin{aligned} a^{-1}f_0(\varphi(x)) - a^{-1}f_0(\varphi(x)+a) &= a^{-1}f_0(\varphi(x))\\ &= a^{-1}a^{n+1}u(\varphi^{-(n+1)}(\varphi(x)))\\ &= a^{-n}u(\varphi^{-n}(x)) = f_0(x).\end{aligned}$$

Furthemore, if f_0 is continuous then so is u and (22) holds. Assume now u is continuous and (22) holds. It is easy to see that then $f_0\,|_{[x_0,1-a)}$ is continuous and u is bounded, say $|\,u(x)\,|\leq M$ for $x\in[x_0,x_1)$, whence $|\,f_0(x)\,|\leq a^n M$ for $x\in[x_n,x_{n+1}]$, $n\in\mathrm{N}$ and, consequently, $\lim_{x\to 1-a} f(x)=0$. This proves that the function f_0 is continuous iff u is continuous and (22) holds. Now define $f_n:[x_n,+\infty)\to\mathbb{R}$ for negative integers n by

$$f_n(x):=\begin{cases} f_{n+1}(x), & x\in[x_{n+1},+\infty),\\ a^{-1}f_{n+1}(\varphi(x))-a^{-1}f_{n+1}(\varphi(x)+a), & x\in[x_n,x_{n+1}),\end{cases}$$

and observe that if for some negative integer n the function f_{n+1} is a continuous solution of (23) then f_n does. Hence we can define a function $f:(0,+\infty)\to\mathbb{R}$ by

$$f:=f_0\cup f_{-1}\cup f_{-2}\cup\dots.$$

This function is a solution of (23), and so of (18), an extension of u, and f is continuous iff f_0 does. Moreover, (21) holds as f vanishes on $[1-a,+\infty)$. Finally, if $\tilde{f}$ is an extension of u to a solution of (18) such that $\lim_{x\to+\infty}x\tilde{f}(x)=0$ then applying Theorem 7 and an induction we see that $\tilde{f}$ coincides with f_n on $[x_n,+\infty)$ for non-positive integers n whence $\tilde{f}=f$.

□

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